CHAPTER 1

# Rational Numbers

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## 1.1 Introduction

In Mathematics, we frequently come across simple equations to be solved. For example, the equation

$$x + 2 = 13 \qquad (1)$$

is solved when $x = 11$, because this value of $x$ satisfies the given equation. The solution 11 is a **natural number**. On the other hand, for the equation

$$x + 5 = 5 \qquad (2)$$

the solution gives the **whole number** 0 (zero). If we consider only natural numbers, equation (2) cannot be solved. To solve equations like (2), we added the number zero to the collection of natural numbers and obtained the whole numbers. Even whole numbers will not be sufficient to solve equations of type

$$x + 18 = 5 \qquad (3)$$

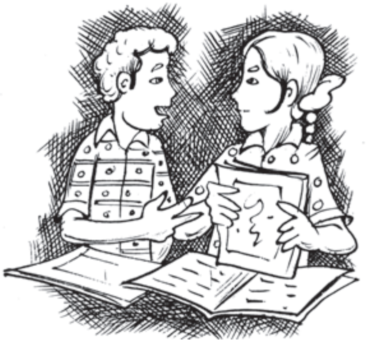
Do you see 'why'? We require the number –13 which is not a whole number. This led us to think of **integers, (positive and negative)**. Note that the positive integers correspond to natural numbers. One may think that we have enough numbers to solve all simple equations with the available list of integers. Now consider the equations

$$2x = 3 \qquad (4)$$

$$5x + 7 = 0 \qquad (5)$$

for which we cannot find a solution from the integers. (Check this)

We need the numbers $\frac{3}{2}$ to solve equation (4) and $\frac{-7}{5}$ to solve equation (5). This leads us to the collection of **rational numbers**.



We have already seen basic operations on rational numbers. We now try to explore some properties of operations on the different types of numbers seen so far.

## 1.2 Properties of Rational Numbers

### 1.2.1 Closure

**(i) Whole numbers**

Let us revisit the closure property for all the operations on whole numbers in brief.

| Operation | Numbers | Remarks |
|---|---|---|
| Addition | $0 + 5 = 5$, a whole number<br>$4 + 7 = \ldots$ . Is it a whole number?<br>In general, $a + b$ is a whole number for any two whole numbers $a$ and $b$. | Whole numbers are closed under addition. |
| Subtraction | $5 - 7 = -2$, which is not a whole number. | Whole numbers are **not** closed under subtraction. |
| Multiplication | $0 \times 3 = 0$, a whole number<br>$3 \times 7 = \ldots$ . Is it a whole number?<br>In general, if $a$ and $b$ are any two whole numbers, their product $ab$ is a whole number. | Whole numbers are closed under multiplication. |
| Division | $5 \div 8 = \frac{5}{8}$, which is not a whole number. | Whole numbers are **not** closed under division. |

Check for closure property under all the four operations for natural numbers.

**(ii) Integers**

Let us now recall the operations under which integers are closed.

| Operation | Numbers | Remarks |
|---|---|---|
| Addition | $-6 + 5 = -1$, an integer<br>Is $-7 + (-5)$ an integer?<br>Is $8 + 5$ an integer?<br>In general, $a + b$ is an integer for any two integers $a$ and $b$. | Integers are closed under addition. |
| Subtraction | $7 - 5 = 2$, an integer<br>Is $5 - 7$ an integer?<br>$-6 - 8 = -14$, an integer | Integers are closed under subtraction. |

| | | |
|---|---|---|
| | $-6-(-8)=2$, an integer<br>Is $8-(-6)$ an integer?<br>In general, for any two integers $a$ and $b$, $a-b$ is again an integer. Check if $b-a$ is also an integer. | |
| Multiplication | $5 \times 8 = 40$, an integer<br>Is $-5 \times 8$ an integer?<br>$-5 \times (-8) = 40$, an integer<br>In general, for any two integers $a$ and $b$, $a \times b$ is also an integer. | Integers are closed under multiplication. |
| Division | $5 \div 8 = \frac{5}{8}$, which is not an integer. | Integers are **not** closed under division. |

You have seen that whole numbers are closed under addition and multiplication but not under subtraction and division. However, integers are closed under addition, subtraction and multiplication but not under division.

**(iii) Rational numbers**

Recall that a number which can be written in the form $\frac{p}{q}$, where $p$ and $q$ are integers and $q \neq 0$ is called a **rational number**. For example, $-\frac{2}{3}, \frac{6}{7}, \frac{9}{-5}$ are all rational numbers. Since the numbers 0, –2, 4 can be written in the form $\frac{p}{q}$, they are also rational numbers. (Check it!)

(a) You know how to add two rational numbers. Let us add a few pairs.

$$\frac{3}{8}+\frac{(-5)}{7}=\frac{21+(-40)}{56}=\frac{-19}{56} \quad \text{(a rational number)}$$

$$\frac{-3}{8}+\frac{(-4)}{5}=\frac{-15+(-32)}{40}=... \quad \text{Is it a rational number?}$$

$$\frac{4}{7}+\frac{6}{11}=... \quad \text{Is it a rational number?}$$

We find that sum of two rational numbers is again a rational number. Check it for a few more pairs of rational numbers.

We say that *rational numbers are closed under addition. That is, for any two rational numbers a and b, a + b is also a rational number*.

(b) Will the difference of two rational numbers be again a rational number?

We have,

$$\frac{-5}{7}-\frac{2}{3}=\frac{-5 \times 3 - 2 \times 7}{21}=\frac{-29}{21} \quad \text{(a rational number)}$$

$$\frac{5}{8} - \frac{4}{5} = \frac{25-32}{40} = \ldots$$ Is it a rational number?

$$\frac{3}{7} - \left(\frac{-8}{5}\right) = \ldots$$ Is it a rational number?

Try this for some more pairs of rational numbers. We find that *rational numbers are closed under subtraction. That is, for any two rational numbers a and b, a – b is also a rational number*.

(c) Let us now see the product of two rational numbers.

$$\frac{-2}{3} \times \frac{4}{5} = \frac{-8}{15};\ \frac{3}{7} \times \frac{2}{5} = \frac{6}{35}$$ (both the products are rational numbers)

$$-\frac{4}{5} \times \frac{-6}{11} = \ldots$$ Is it a rational number?

Take some more pairs of rational numbers and check that their product is again a rational number.

We say that *rational numbers are closed under multiplication. That is, for any two rational numbers a and b, a × b is also a rational number*.

(d) We note that $\frac{-5}{3} \div \frac{2}{5} = \frac{-25}{6}$ (a rational number)

$\frac{2}{7} \div \frac{5}{3} = \ldots$ . Is it a rational number? $\frac{-3}{8} \div \frac{-2}{9} = \ldots$ . Is it a rational number?

Can you say that rational numbers are closed under division?

We find that for any rational number $a$, $a \div 0$ is **not defined**.

So rational numbers are **not closed** under division.

However, if we exclude zero then the collection of, all other rational numbers is closed under division.

## TRY THESE

Fill in the blanks in the following table.

| Numbers | Closed under | | | |
|---|---|---|---|---|
| | **addition** | **subtraction** | **multiplication** | **division** |
| Rational numbers | Yes | Yes | ... | No |
| Integers | ... | Yes | ... | No |
| Whole numbers | ... | ... | Yes | ... |
| Natural numbers | ... | No | ... | ... |

### 1.2.2 Commutativity

**(i) Whole numbers**

Recall the commutativity of different operations for whole numbers by filling the following table.

| Operation | Numbers | Remarks |
|---|---|---|
| Addition | $0 + 7 = 7 + 0 = 7$<br>$2 + 3 = ... + ... = ....$<br>For any two whole numbers $a$ and $b$,<br>$a + b = b + a$ | Addition is commutative. |
| Subtraction | ......... | Subtraction is not commutative. |
| Multiplication | ......... | Multiplication is commutative. |
| Division | ......... | Division is not commutative. |

Check whether the commutativity of the operations hold for natural numbers also.

**(ii) Integers**

Fill in the following table and check the commutativity of different operations for integers:

| Operation | Numbers | Remarks |
|---|---|---|
| Addition | ......... | Addition is commutative. |
| Subtraction | Is $5 - (-3) = -3 - 5$? | Subtraction is not commutative. |
| Multiplication | ......... | Multiplication is commutative. |
| Division | ......... | Division is not commutative. |

**(iii) Rational numbers**

**(a) Addition**

You know how to add two rational numbers. Let us add a few pairs here.

$$\frac{-2}{3}+\frac{5}{7}=\frac{1}{21} \text{ and } \frac{5}{7}+\left(\frac{-2}{3}\right)=\frac{1}{21}$$

So, $$\frac{-2}{3}+\frac{5}{7}=\frac{5}{7}+\left(\frac{-2}{3}\right)$$

Also, $$\frac{-6}{5}+\left(\frac{-8}{3}\right) = ... \text{ and } \frac{-8}{3}+\left(\frac{-6}{5}\right)=...$$

Is $$\frac{-6}{5}+\left(\frac{-8}{3}\right)=\left(\frac{-8}{3}\right)+\left(\frac{-6}{5}\right)?$$

Is $\frac{-3}{8}+\frac{1}{7}=\frac{1}{7}+\left(\frac{-3}{8}\right)$?

You find that two *rational numbers can be added in any order. We say that addition is commutative for rational numbers. That is, for any two rational numbers a and b, a + b = b + a.*

**(b) Subtraction**

Is $\frac{2}{3}-\frac{5}{4}=\frac{5}{4}-\frac{2}{3}$?

Is $\frac{1}{2}-\frac{3}{5}=\frac{3}{5}-\frac{1}{2}$?

You will find that subtraction is not commutative for rational numbers.

Note that subtraction is not commutative for integers and integers are also rational numbers. So, subtraction will not be commutative for rational numbers too.

**(c) Multiplication**

We have, $\frac{-7}{3}\times\frac{6}{5}=\frac{-42}{15}=\frac{6}{5}\times\left(\frac{-7}{3}\right)$

Is $\frac{-8}{9}\times\left(\frac{-4}{7}\right)=\frac{-4}{7}\times\left(\frac{-8}{9}\right)$?

Check for some more such products.

You will find that *multiplication is commutative for rational numbers*.

*In general, a × b = b × a for any two rational numbers a and b.*

**(d) Division**

Is $\frac{-5}{4}\div\frac{3}{7}=\frac{3}{7}\div\left(\frac{-5}{4}\right)$?

You will find that expressions on both sides are not equal.

So division is **not commutative** for rational numbers.

**TRY THESE**

Complete the following table:

| Numbers | Commutative for | | | |
|---|---|---|---|---|
| | **addition** | **subtraction** | **multiplication** | **division** |
| Rational numbers | Yes | ... | ... | ... |
| Integers | ... | No | ... | ... |
| Whole numbers | ... | ... | Yes | ... |
| Natural numbers | ... | ... | ... | No |

### 1.2.3 Associativity

**(i) Whole numbers**

Recall the associativity of the four operations for whole numbers through this table:

| Operation | Numbers | Remarks |
|---|---|---|
| Addition | ......... | Addition is associative |
| Subtraction | ......... | Subtraction is **not** associative |
| Multiplication | Is $7 \times (2 \times 5) = (7 \times 2) \times 5$?<br>Is $4 \times (6 \times 0) = (4 \times 6) \times 0$?<br>For any three whole numbers $a$, $b$ and $c$<br>$a \times (b \times c) = (a \times b) \times c$ | Multiplication is associative |
| Division | ......... | Division is **not** associative |

Fill in this table and verify the remarks given in the last column.
Check for yourself the associativity of different operations for natural numbers.

**(ii) Integers**

Associativity of the four operations for integers can be seen from this table

| Operation | Numbers | Remarks |
|---|---|---|
| Addition | Is $(-2) + [3 + (-4)]$<br>$= [(-2) + 3)] + (-4)$?<br>Is $(-6) + [(-4) + (-5)]$<br>$= [(-6) + (-4)] + (-5)$?<br>For any three integers $a$, $b$ and $c$<br>$a + (b + c) = (a + b) + c$ | Addition is associative |
| Subtraction | Is $5 - (7 - 3) = (5 - 7) - 3$? | Subtraction is **not** associative |
| Multiplication | Is $5 \times [(-7) \times (-8)$<br>$= [5 \times (-7)] \times (-8)$?<br>Is $(-4) \times [(-8) \times (-5)]$<br>$= [(-4) \times (-8)] \times (-5)$?<br>For any three integers $a$, $b$ and $c$<br>$a \times (b \times c) = (a \times b) \times c$ | Multiplication is associative |
| Division | Is $[(-10) \div 2] \div (-5)$<br>$= (-10) \div [2 \div (-5)]$? | Division is **not** associative |

**(iii) Rational numbers**

**(a) Addition**

We have $\frac{-2}{3}+\left[\frac{3}{5}+\left(\frac{-5}{6}\right)\right]=\frac{-2}{3}+\left(\frac{-7}{30}\right)=\frac{-27}{30}=\frac{-9}{10}$

$$\left[\frac{-2}{3}+\frac{3}{5}\right]+\left(\frac{-5}{6}\right)=\frac{-1}{15}+\left(\frac{-5}{6}\right)=\frac{-27}{30}=\frac{-9}{10}$$

So, $\frac{-2}{3}+\left[\frac{3}{5}+\left(\frac{-5}{6}\right)\right]=\left[\frac{-2}{3}+\frac{3}{5}\right]+\left(\frac{-5}{6}\right)$

Find $\frac{-1}{2}+\left[\frac{3}{7}+\left(\frac{-4}{3}\right)\right]$ and $\left[\frac{-1}{2}+\frac{3}{7}\right]+\left(\frac{-4}{3}\right)$. Are the two sums equal?

Take some more rational numbers, add them as above and see if the two sums are equal. We find that *addition is associative for rational numbers. That is, for any three rational numbers a, b* and *c*, $a + (b + c) = (a + b) + c$.

**(b) Subtraction**

You already know that subtraction is not associative for integers, then what about rational numbers.

Is $\frac{-2}{3}-\left[\frac{-4}{5}-\frac{1}{2}\right]=\left[\frac{2}{3}-\left(\frac{-4}{5}\right)\right]-\frac{1}{2}$?

Check for yourself.

Subtraction is **not associative** for rational numbers.

**(c) Multiplication**

Let us check the associativity for multiplication.

$$\frac{-7}{3}\times\left(\frac{5}{4}\times\frac{2}{9}\right)=\frac{-7}{3}\times\frac{10}{36}=\frac{-70}{108}=\frac{-35}{54}$$

$$\left(\frac{-7}{3}\times\frac{5}{4}\right)\times\frac{2}{9}=\ldots$$

We find that $\frac{-7}{3}\times\left(\frac{5}{4}\times\frac{2}{9}\right)=\left(\frac{-7}{3}\times\frac{5}{4}\right)\times\frac{2}{9}$

Is $\frac{2}{3}\times\left(\frac{-6}{7}\times\frac{4}{5}\right)=\left(\frac{2}{3}\times\frac{-6}{7}\right)\times\frac{4}{5}$?

Take some more rational numbers and check for yourself.

We observe that *multiplication is associative for rational numbers. That is for any three rational numbers a, b* and *c*, $a \times (b \times c) = (a \times b) \times c$.

**(d) Division**

Recall that division is not associative for integers, then what about rational numbers?

Let us see if $\frac{1}{2} \div \left[\frac{-1}{3} \div \frac{2}{5}\right] = \left[\frac{1}{2} \div \left(\frac{-1}{3}\right)\right] \div \frac{2}{5}$

We have, LHS $= \frac{1}{2} \div \left(\frac{-1}{3} \div \frac{2}{5}\right) = \frac{1}{2} \div \left(\frac{-1}{3} \times \frac{5}{2}\right)$ (reciprocal of $\frac{2}{5}$ is $\frac{5}{2}$)

$$= \frac{1}{2} \div \left(-\frac{5}{6}\right) = \ldots$$

RHS $= \left[\frac{1}{2} \div \left(\frac{-1}{3}\right)\right] \div \frac{2}{5}$

$$= \left(\frac{1}{2} \times \frac{-3}{1}\right) \div \frac{2}{5} = \frac{-3}{2} \div \frac{2}{5} = \ldots$$

Is LHS = RHS? Check for yourself. You will find that division is **not associative** for rational numbers.

## TRY THESE

Complete the following table:

| Numbers | Associative for | | | |
|---|---|---|---|---|
| | **addition** | **subtraction** | **multiplication** | **division** |
| Rational numbers | ... | ... | ... | No |
| Integers | ... | ... | Yes | ... |
| Whole numbers | Yes | ... | ... | ... |
| Natural numbers | ... | No | ... | ... |

**Example 1:** Find $\frac{3}{7} + \left(\frac{-6}{11}\right) + \left(\frac{-8}{21}\right) + \left(\frac{5}{22}\right)$

**Solution:** $\frac{3}{7} + \left(\frac{-6}{11}\right) + \left(\frac{-8}{21}\right) + \left(\frac{5}{22}\right)$

$$= \frac{198}{462} + \left(\frac{-252}{462}\right) + \left(\frac{-176}{462}\right) + \left(\frac{105}{462}\right)$$ (Note that 462 is the LCM of 7, 11, 21 and 22)

$$= \frac{198 - 252 - 176 + 105}{462} = \frac{-125}{462}$$

We can also solve it as.

$$\frac{3}{7}+\left(\frac{-6}{11}\right)+\left(\frac{-8}{21}\right)+\frac{5}{22}$$

$$=\left[\frac{3}{7}+\left(\frac{-8}{21}\right)\right]+\left[\frac{-6}{11}+\frac{5}{22}\right]$$ (by using commutativity and associativity)

$$=\left[\frac{9+(-8)}{21}\right]+\left[\frac{-12+5}{22}\right]$$ (LCM of 7 and 21 is 21; LCM of 11 and 22 is 22)

$$=\frac{1}{21}+\left(\frac{-7}{22}\right)=\frac{22-147}{462}=\frac{-125}{462}$$

Do you think the properties of commutativity and associativity made the calculations easier?

**Example 2:** Find $\frac{-4}{5}\times\frac{3}{7}\times\frac{15}{16}\times\left(\frac{-14}{9}\right)$

**Solution:** We have

$$\frac{-4}{5}\times\frac{3}{7}\times\frac{15}{16}\times\left(\frac{-14}{9}\right)$$

$$=\left(-\frac{4\times 3}{5\times 7}\right)\times\left(\frac{15\times(-14)}{16\times 9}\right)$$

$$=\frac{-12}{35}\times\left(\frac{-35}{24}\right)=\frac{-12\times(-35)}{35\times 24}=\frac{1}{2}$$

We can also do it as.

$$\frac{-4}{5}\times\frac{3}{7}\times\frac{15}{16}\times\left(\frac{-14}{9}\right)$$

$$=\left(\frac{-4}{5}\times\frac{15}{16}\right)\times\left[\frac{3}{7}\times\left(\frac{-14}{9}\right)\right]$$ (Using commutativity and associativity)

$$=\frac{-3}{4}\times\left(\frac{-2}{3}\right)=\frac{1}{2}$$

### 1.2.4 The role of zero (0)

Look at the following.

$2 + 0 = 0 + 2 = 2$ (Addition of 0 to a whole number)

$-5 + 0 = \ldots + \ldots = -5$ (Addition of 0 to an integer)

$\frac{-2}{7} + \ldots = 0 + \left(\frac{-2}{7}\right) = \frac{-2}{7}$ (Addition of 0 to a rational number)

You have done such additions earlier also. Do a few more such additions.

What do you observe? You will find that when you add 0 to a whole number, the sum is again that whole number. This happens for integers and rational numbers also.

In general, $a + 0 = 0 + a = a$, where $a$ is a whole number

$b + 0 = 0 + b = b$, where $b$ is an integer

$c + 0 = 0 + c = c$, where $c$ is a rational number

*Zero is called the identity for the addition of rational numbers. It is the additive identity for integers and whole numbers as well.*

### 1.2.5 The role of 1

We have,

$$5 \times 1 = 5 = 1 \times 5 \quad \text{(Multiplication of 1 with a whole number)}$$

$$\frac{-2}{7} \times 1 = \ldots \times \ldots = \frac{-2}{7}$$

$$\frac{3}{8} \times \ldots = 1 \times \frac{3}{8} = \frac{3}{8}$$

What do you find?

You will find that when you multiply any rational number with 1, you get back the same rational number as the product. Check this for a few more rational numbers. You will find that, $a \times 1 = 1 \times a = a$ for any rational number $a$.

We say *that* 1 *is the multiplicative identity for rational numbers.*

Is 1 the multiplicative identity for integers? For whole numbers?

## THINK, DISCUSS AND WRITE

If a property holds for rational numbers, will it also hold for integers? For whole numbers? Which will? Which will not?

### 1.2.6 Distributivity of multiplication over addition for rational numbers

To understand this, consider the rational numbers $\frac{-3}{4}, \frac{2}{3}$ and $\frac{-5}{6}$.

$$\frac{-3}{4} \times \left\{\frac{2}{3} + \left(\frac{-5}{6}\right)\right\} = \frac{-3}{4} \times \left\{\frac{(4) + (-5)}{6}\right\}$$

$$= \frac{-3}{4} \times \left(\frac{-1}{6}\right) = \frac{3}{24} = \frac{1}{8}$$

Also $$\frac{-3}{4} \times \frac{2}{3} = \frac{-3 \times 2}{4 \times 3} = \frac{-6}{12} = \frac{-1}{2}$$

And $\frac{-3}{4} \times \frac{-5}{6} = \frac{5}{8}$

Therefore $\left(\frac{-3}{4} \times \frac{2}{3}\right) + \left(\frac{-3}{4} \times \frac{-5}{6}\right) = \frac{-1}{2} + \frac{5}{8} = \frac{1}{8}$

Thus, $\frac{-3}{4} \times \left\{\frac{2}{3} + \frac{-5}{6}\right\} = \left(\frac{-3}{4} \times \frac{2}{3}\right) + \left(\frac{-3}{4} \times \frac{-5}{6}\right)$

**Distributivity of Multiplication over Addition and Subtraction.**
For all rational numbers $a$, $b$ and $c$,
$a\,(b + c) = ab + ac$
$a\,(b - c) = ab - ac$

## TRY THESE

Find using distributivity. (i) $\left\{\frac{7}{5} \times \left(\frac{-3}{12}\right)\right\} + \left\{\frac{7}{5} \times \frac{5}{12}\right\}$ (ii) $\left\{\frac{9}{16} \times \frac{4}{12}\right\} + \left\{\frac{9}{16} \times \frac{-3}{9}\right\}$

**Example 3:** Find $\frac{2}{5} \times \frac{-3}{7} - \frac{1}{14} - \frac{3}{7} \times \frac{3}{5}$

**Solution:** $\frac{2}{5} \times \frac{-3}{7} - \frac{1}{14} - \frac{3}{7} \times \frac{3}{5} = \frac{2}{5} \times \frac{-3}{7} - \frac{3}{7} \times \frac{3}{5} - \frac{1}{14}$ (by commutativity)

$= \frac{2}{5} \times \frac{-3}{7} + \left(\frac{-3}{7}\right) \times \frac{3}{5} - \frac{1}{14}$

$= \frac{-3}{7}\left(\frac{2}{5} + \frac{3}{5}\right) - \frac{1}{14}$ (by distributivity)

$= \frac{-3}{7} \times 1 - \frac{1}{14} = \frac{-6-1}{14} = \frac{-1}{2}$

## EXERCISE 1.1

1. Name the property under multiplication used in each of the following.

   (i) $\frac{-4}{5} \times 1 = 1 \times \frac{-4}{5} = -\frac{4}{5}$ (ii) $-\frac{13}{17} \times \frac{-2}{7} = \frac{-2}{7} \times \frac{-13}{17}$

   (iii) $\frac{-19}{29} \times \frac{29}{-19} = 1$

2. Tell what property allows you to compute $\frac{1}{3} \times \left(6 \times \frac{4}{3}\right)$ as $\left(\frac{1}{3} \times 6\right) \times \frac{4}{3}$.

3. The product of two rational numbers is always a ________.

## WHAT HAVE WE DISCUSSED?

1. Rational numbers are **closed** under the operations of addition, subtraction and multiplication.
2. The operations addition and multiplication are
   (i) **commutative** for rational numbers.
   (ii) **associative** for rational numbers.
3. The rational number 0 is the **additive identity** for rational numbers.
4. The rational number 1 is the **multiplicative identity** for rational numbers.
5. **Distributivity** of rational numbers: For all rational numbers $a$, $b$ and $c$,
   $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$
6. Between any two given rational numbers there are countless rational numbers. The idea of **mean** helps us to find rational numbers between two rational numbers.

## NOTES